

Relativistic elastomechanics as a lagrangian field theory

Jerzy Kijowski

*Institute for Theoretical Physics, Polish Academy of Sciences,
Al. Lotników 32/46, 02-668 Warsaw, Poland*

Giulio Magli

Department of Mathematics, University of Milan, Via Saldini 50, 20133 Milan, Italy

Received 20 May 1991
(Revised 1 October 1991)

A new formulation of relativistic elastomechanics is presented. It is free of any assumption about the existence of a global relaxation state of the material. The strain, the stress and the energy–momentum tensors are expressed in terms of the first-order derivatives of a field describing the configuration of the material. Its elastic properties are completely determined by a scalar function describing the dependence of the mechanical energy accumulated in the deformed material upon the three invariants of the deformation. The stress–strain relations are generated in a canonical way by this function. Dynamical equations of the theory are derived from the variational principle. They form a hyperbolic system of second-order partial differential equations for the unknown field. Energy–momentum conservation laws are consequences of the Noether theorem. The hamiltonian formulation of the dynamics and the linear version of the theory are also discussed.

*Keywords: finite elasticity, lagrangian field theory
1991 MSC: 73B05*

1. Introduction

Classical, non-relativistic elasticity is a good example of a lagrangian, hyperbolic field theory. The configuration of an elastic body is described at each instant of time by a three-component field called a displacement vector. The local deformation of the material is described by the *strain tensor* defined as a combination of the first derivatives of the field. The time-dependent configuration has to satisfy a system of second-order hyperbolic partial differential equations [1,2]. These dynamical equations can be derived from the first-order variational formula and the energy–momentum conservation laws follow from the invariance of the lagrangian via the standard procedure based on the Noether theorem.

The special feature of this theory is that the dynamical equations are actually

equivalent to the conservation laws if the energy–momentum tensor is expressed in terms of the first derivatives of the unknown field via the stress–strain relations. This observation was a starting point of many attempts to formulate the relativistic version of elasticity theory [3–7]. In fact, it is easy to generalize the energy–momentum conservation laws to the fully relativistic situation. In this way we obtain the first-order differential equations for the energy–momentum tensor [eq. (35) of the present paper] which have been proposed by all the above authors. The trouble is, however, that the theory is not closed if we are not able to express the energy–momentum tensor in terms of the first derivatives of the field describing the configuration of the material. This is the reason why none of these theories has been consistently formulated as a hyperbolic field theory.

The problem consists therefore in finding the relativistic description of the configuration of the material. In a generic, curved space–time there is no analogue of a displacement vector. Some authors propose to describe a configuration of an elastic body at a given instant of time as a mapping between the actual distribution and the hypothetical “equilibrium distribution” of the matter in the three-dimensional physical space [8,9]. A diffeomorphism between three-dimensional manifolds is again (at least formally) a three-component field and it is possible to formulate in this way a general-relativistic version of the theory of elasticity. The assumption about the existence of a global equilibrium configuration is, however, poorly justified from the physical point of view. Moreover, it violates the relativistic invariance of the theory.

A simple way to overcome this difficulty is to introduce an abstract, three-dimensional “material space” as a collection of all “particles” of the idealized material [10–12] and to treat consequently “material coordinates” as field variables of the theory. In the present paper we propose a simple formulation of relativistic elasticity theory based on this approach. It is fully relativistic, intimately non-linear, hyperbolic and lagrangian. Our theory is a straightforward continuation of the formulation of hydrodynamics given by one of us [13–15]. The difference between a fluid and an elastic body consists only in different constitutive relations, but the entire dynamical structure of both theories is the same.

We show in the last part of the paper that the linearized version of our field equations coincides with the equations proposed by some authors (see refs. [4,16,17]). However, relativistic elasticity is important not only because of its linearized version, which can be used for the description of gravitational wave detectors and generators. Its theoretical importance lies in the fact that it provides a simple, self-consistent model of stable relativistic matter and we hope it to be useful in the description of stellar matter in many astrophysical problems. A first application of the present formulation to the relativistic equilibrium equations of a non-rotating star will soon be published [18].

2. Kinematics

Let \mathcal{M} be the general-relativistic space–time equipped with the fixed pseudo-riemannian metric tensor $g_{\mu\nu}$ ($\mu, \nu=0, 1, 2, 3$), whose signature is $(-, +, +, +)$. Denote by Z the collection of all idealized “molecules” of the material organized in an abstract three-dimensional manifold called the *material space*. The space–time configuration of the material is completely described if a mapping

$$\mathcal{G}: \mathcal{M} \rightarrow Z$$

is specified; it assigns to a given point of the physical space–time (i.e. to a given point of the space and a given instant of time) the “molecule” of the material which passes through this particular point at this particular time. Generically, the mapping \mathcal{G} is constant on one-dimensional submanifolds (curves) in \mathcal{M} which correspond to the world lines of the molecules. We assume these lines to be timelike. It is easy to prove that the dynamics of the theory (section 5) prevents the lines from becoming spacelike.

Given a coordinate system (ξ^a) ($a=1, 2, 3$) in Z and a coordinate system (x^μ) in \mathcal{M} , the configuration of the material is described by three fields $\xi^a = \xi^a(x^\mu)$. The physical laws describing the mechanical properties of the elastic material will be formulated in terms of a system of second-order hyperbolic partial differential equations for the three unknown fields ξ^a .

The tangent mapping

$$\mathcal{G}_* : T_x \mathcal{M} \mapsto T_{\xi(x)} Z$$

is uniquely described by the 3×4 matrix composed of partial derivatives of the fields, $(\xi^a{}_\mu) := (\partial_\mu \xi^a)$, and may be called the *relativistic deformation gradient*. Our assumption about the character of world lines means that the matrix has maximal rank and that its kernel (i.e. the collection of vectors u^μ satisfying the equation $u^\mu \xi^a{}_\mu = 0$) is a one-dimensional timelike subspace of the tangent space $T_x \mathcal{M}$. If, moreover, the space–time is temporally oriented we may uniquely choose u^μ in such a way that it belongs to the future light cone and is normalized. This vector is called the *velocity field* of the matter. Its components u^μ are uniquely given by the following conditions:

$$u^\mu \xi^a{}_\mu = 0, \quad g_{\mu\nu} u^\mu u^\nu = u^\mu u_\mu = -1, \quad u^0 > 0. \quad (1)$$

Solving them, we may express u^μ as a non-linear function of the components $\xi^a{}_\mu$ of the deformation gradient and of the space–time metric $g_{\mu\nu}$.

Given the space–time configuration of the matter $\xi^a = \xi^a(x^\mu)$ we may thus decompose at each point $x \in \mathcal{M}$ the tangent space $T_x \mathcal{M}$ into its parts tangent and orthogonal to the velocity field. The corresponding projector operators are defined as

$$U_\nu^\mu := -u^\mu u_\nu, \quad E_\nu^\mu := \delta_\nu^\mu + u^\mu u_\nu.$$

3. Geometric structure of the material space

We assume the material space Z to be equipped with a riemannian (positive) metric γ_{ab} called the *material metric*. The metric is “frozen” in the material and is not a dynamical object of the theory. To understand the physical meaning of the material metric we assume that each “infinitesimally” small portion of the material will tend spontaneously to the relaxed state when no external forces act on it. The metric γ describes the space distances between neighbouring “molecules”, calculated in the relaxed state of the material, with respect to the rest frame. To measure the components $\gamma_{ab}(\xi)$ in this way we have to relax the material at different points ξ separately since global relaxation of the material may not be possible. This means that the space Z may not be isometric with any three-dimensional subspace of the space–time. The same phenomenon occurs in classical (non-linear) elastomechanics, when the material exhibits internal stresses frozen in it. In this case no global relaxation is possible and the structure of the material has to be described with a curved material metric, even if the physical space is flat.

From the mathematical point of view, the components $\gamma_{ab} = \gamma_{ab}(\xi)$ should be considered as given functions. They describe axiomatically the properties of the material we consider. The theory is, however, fully invariant with respect to reparametrizations of the material space. This means that also classical, non-relativistic elasticity may be formulated in terms of curvilinear coordinates, which make the dependence $\gamma_{ab}(\xi)$ highly non-trivial, even if the material metric is flat.

The metric structure of the material space enables us to introduce a volume structure in Z ,

$$\omega = \rho_0 \sqrt{\det \gamma} d\xi^1 \wedge d\xi^2 \wedge d\xi^3, \quad (2)$$

where $\rho_0 = \rho_0(\xi)$ is given axiomatically as the density of the material (moles per unit volume of the relaxed material). In most applications the material is homogeneous and ρ_0 is constant. Denoting $h = \rho_0 \sqrt{\det \gamma}$, formula (2) becomes identical with formula (16) of ref. [14] and the entire construction of the matter current given below is therefore identical with the corresponding construction for hydrodynamics.

The pull-back of the volume form ω to the space–time is a differential three-form in the four-dimensional manifold \mathcal{M} , i.e. a vector density. We denote it by J and call it the *matter current*. To calculate its components we substitute in formula (2) the coordinates ξ^a by the values of the field configurations treated as functions on space–time,

$$\begin{aligned} J &:= \mathcal{G}^* \omega = \rho_0 \sqrt{\det \gamma} d\xi^1(x) \wedge d\xi^2(x) \wedge d\xi^3(x) \\ &= \rho_0 \sqrt{\det \gamma} \xi^1_{\nu} \xi^2_{\mu} \xi^3_{\sigma} dx^{\nu} \wedge dx^{\mu} \wedge dx^{\sigma} = J^{\mu} (\partial_{\mu} \rfloor dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3), \quad (3) \\ J^{\mu} &= \rho_0 \sqrt{\det \gamma} \epsilon^{\mu\nu\rho\sigma} \xi^1_{\nu} \xi^2_{\rho} \xi^3_{\sigma}. \end{aligned}$$

The matter current is a priori conserved due to its geometric construction. Indeed, the exterior derivative of J is equal to the pull-back of the exterior derivative of ω . It vanishes identically being a four-form in the three-dimensional space Z ,

$$(\partial_\mu J^\mu) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = dJ = d(\mathcal{G}^* \omega) = \mathcal{G}^*(d\omega) = 0, \tag{4}$$

or, equivalently,

$$\partial_\mu J^\mu = 0. \tag{5}$$

The last identity may also be calculated directly from formula (3) without any reference to the geometric proof (4).

Let us observe that $J^\mu \xi^a{}_\mu = 0$, since $\epsilon^{\mu\nu\rho\sigma} \xi^1{}_\nu \xi^2{}_\rho \xi^3{}_\sigma \xi^a{}_\mu$ is the determinant of a matrix with two identical columns. This means that J^μ is proportional to the velocity field,

$$J^\mu = \rho \sqrt{-g} u^\mu, \tag{6}$$

where the scalar function $\rho = (g^{-1} J^\mu J_\mu)^{1/2}$, with J^μ given by (3), is a non-linear function of the components of the deformation gradient. Its physical meaning is the actual, rest frame matter density.

4. Relativistic strain tensor

Given the space-time configuration of the elastic material, we define the pull-back of the material metric γ as $h := \mathcal{G}^* \gamma$, or, in terms of coordinates ^{#1}

$$h_{\mu\nu} = \gamma_{ab} \xi^a{}_\mu \xi^b{}_\nu. \tag{7}$$

The tensor h is obviously symmetric and orthogonal to the velocity field due to the relation

$$h_{\mu\nu} u^\nu = \gamma_{ab} \xi^a{}_\mu \xi^b{}_\nu u^\nu = 0.$$

We define the following symmetric operator:

$$K := g^{-1} h + U,$$

whose components are

$$K^\mu{}_\nu = g^{\mu\rho} h_{\rho\nu} - u^\mu u_\nu.$$

^{#1} This approach is similar to the one used in ref. [12]. However, those authors rather prefer the “push-forward” of the physical metric to the material space, defining in this way the relativistic *Piola* strain tensor. This terminology is slightly misleading since the information about the deformation of the material is actually carried by the difference between two metric tensors (the material one and the physical one) and not by only one of them.

Observe that the velocity vector is an eigenvector of K with eigenvalue 1,

$$K_{\nu}^{\mu} u^{\nu} = u^{\mu}. \quad (8)$$

If the coordinates (x^{μ}) are chosen in such a way that they correspond to the local rest frame at the point in which we calculate K , its components are given by

$$(K_{\nu}^{\mu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & (g^{mk}h_{kl}) & & \\ 0 & & & \end{pmatrix}. \quad (9)$$

This proves that K is positive definite. If, moreover, the material is relaxed at this particular point, the material metric coincides with the physical metric. This means that the two positive metric tensors h_{kl} and g_{kl} ($k, l = 1, 2, 3$) coincide and we have $g^{km}h_{ml} = \delta_l^k$. The operator K becomes therefore the identity operator if and only if the material is locally relaxed, and the deformation consists in the fact that K is different from unity. Any function of K which vanishes on the identity operator may be used to measure the deformation. A possible choice could be simply $\frac{1}{2}(I - K)$ (similar to "the relativistic lagrangian strain tensor" proposed by G.A. Maugin). We prefer, however, another definition of the strain tensor which makes the theory (and especially the stress-strain relations) particularly elegant #2:

$$S := -\frac{1}{2} \log K. \quad (10)$$

We call the symmetric operator S the *relativistic strain tensor*. It measures the *relative* changes of the dimensions of the material along the principal axes of the deformation and is obviously dimensionless.

Due to formula (8), the velocity vector is an eigenvector of any operator-valued function of K . So we have

$$(K^{-1})_{\nu}^{\mu} u^{\nu} = u^{\mu}, \quad (11)$$

$$S_{\nu}^{\mu} u^{\nu} = 0. \quad (12)$$

The tensor S contains the complete information about the local state of the

#2 Another nice feature of our definition is that the strain tensor is not subject to any constraint when the deformation becomes strong. This is not true in the case that the strain tensor is defined as the difference. In this case one has to worry during the evolution that the strain produces a positive operator when added to the identity. Mathematically, it is extremely difficult to handle such constraints. Of course, both definitions coincide in the linear approximation.

matter. In particular, the matter density ρ can also be calculated in terms of S . Indeed, formulae (3) and (6) imply

$$\rho \sqrt{-g} u^0 = J^0 = \rho_0 \sqrt{\det \gamma} \mathcal{D}, \tag{13}$$

where we denoted

$$\mathcal{D} := \epsilon^{0\nu\rho\sigma} \xi^1_\nu \xi^2_\rho \xi^3_\sigma = e^{0hkl} \xi^1_h \xi^2_k \xi^3_l = \epsilon^{hkl} \xi^1_h \xi^2_k \xi^3_l = \det(\xi^a_k). \tag{14}$$

In the rest frame we have $u^0 = 1$ and $\sqrt{-g} = \sqrt{\det g_{kl}}$. Hence, formulae (7), (9) and (13) imply

$$\begin{aligned} \det K &= (\det g^{mk}) (\det h_{kl}) \\ &= (\det g_{mk})^{-1} (\det \gamma) [\det(\xi^a_k)]^2 = \frac{\det \gamma}{-g} \mathcal{D}^2 = \left(\frac{\rho}{\rho_0}\right)^2. \end{aligned}$$

Both the quantities $\det K$ and $(\rho/\rho_0)^2$ are, however, scalars and therefore the identity

$$\det K = (\rho/\rho_0)^2$$

holds in any frame. Taking the trace of (10) we have

$$\text{Tr } S = -\frac{1}{2} \text{Tr } \log K = -\log \sqrt{\det K},$$

which finally implies

$$\rho = \rho_0 \sqrt{\det K} = \rho_0 \exp(-\text{Tr } S). \tag{15}$$

5. Internal energy of elastic deformations

We assume the internal energy of an elastic deformation, accumulated in an infinitesimal portion of the material, to be invariant with respect to its space-time orientation. In this way the internal energy may depend on the three invariants of the strain tensor only (one of the eigenvalues of the strain vanishes a priori). To define an appropriate set of invariants, we decompose the strain tensor into a part proportional to the projector E and a traceless part \tilde{S} :

$$S^\mu_\nu = \tilde{S}^\mu_\nu + \frac{1}{3} S^\rho_\rho E^\mu_\nu, \tag{16}$$

where $\tilde{S}^\mu_\mu = 0$, and introduce the following three invariants:

$$\alpha = \text{Tr } S = S^\mu_\mu, \quad \beta = \frac{1}{2} \text{Tr } \tilde{S}^2 = \frac{1}{2} \tilde{S}^\mu_\nu \tilde{S}^\nu_\mu, \quad \theta = \frac{1}{3} \text{Tr } \tilde{S}^3 = \frac{1}{3} \tilde{S}^\mu_\rho \tilde{S}^\rho_\nu \tilde{S}^\nu_\mu. \tag{17}$$

Formula (15) implies the interpretation of the first invariant as the relative compression of the material:

$$\alpha = -\log(\rho/\rho_0). \tag{18}$$

The second invariant measures the deviation of the deformation from spherical symmetry. The last invariant is of a third order and does not appear in the linear version of the theory; it measures the deviation of the deformation from cylindrical symmetry (see the lecture notes [19]). We assume therefore that the molar internal energy u_I of an elastic deformation depends on the fields ξ^a and their first derivatives only via the above invariants^{#3}. The function $u_I = u_I(\alpha, \beta, \theta)$ has to be considered as given axiomatically. Its choice is equivalent to the choice of the constitutive equations of the material. However, to assure the stability of the model, the function u_I is supposed to vanish and to have a minimum in a locally relaxed state of the matter (i.e., when all the invariants vanish). This means that

$$u_I = \frac{1}{2}A\alpha^2 + B\beta + O(3), \quad (19)$$

where $O(3)$ is a function of (at least) the third order in the strains in the neighbourhood of zero. The simplest model of an elastic material consists in putting this function equal to zero. In this case the energy is a quadratic function of the strain and the constants A and B are related to the usual Lamé coefficients λ and μ by the following relations:

$$\lambda = A - \frac{1}{3}B, \quad \mu = \frac{1}{2}B.$$

We will call this model *quasi-linear*. In general, a non-linear behaviour of the material gives rise to a more complicated functional dependence of the elastic energy. In particular, there is no reason for the independence of u_I from the third-order parameter θ .

The total energy (per mole) is equal to

$$e = m + u_I,$$

where m denotes the molar rest mass of the material (we use geometric units with the velocity of light equal to one). The correspondence principle implies that the lagrangian density of the theory equals

$$\mathcal{L} = -\rho\sqrt{-g}e = -\sqrt{-g}\epsilon, \quad (20)$$

where $\epsilon = \rho e$ denotes the rest frame energy per unit volume. The lagrangian \mathcal{L} has to be considered as a function of the independent variables x^μ , the unknown fields ξ^a and their first derivatives $\xi^a{}_{,\mu}$. The dynamical equations of the theory are obtained from the variational principle $\delta\int\mathcal{L}=0$ and can be written as Euler-Lagrange equations,

$$\partial_\mu \partial\mathcal{L}/\partial\xi^a{}_{,\mu} - \partial\mathcal{L}/\partial\xi^a = 0. \quad (21)$$

The above theory contains hydrodynamics as a particular case when the energy depends on α only [13–15].

^{#3} For an inhomogeneous material we could admit also an a priori dependence of u_I upon the value of the fields ξ^a ; the generalization of our results to this case is obvious.

6. Noether theorem and energy–momentum conservation

The Euler–Lagrange equations (21) can be written as

$$\partial_\mu P_a^\mu = \partial\Lambda/\partial\xi^a, \tag{22}$$

with the canonical momentum P_a^μ defined as usual,

$$P_a^\mu := \partial\Lambda/\partial\xi^a{}_\mu. \tag{23}$$

Geometrically, P_a^μ is a mixed tensor density defined on the cartesian product $\mathcal{M} \times \mathcal{Z}$. It can be called the *relativistic Piola–Kirchhoff momentum tensor*. We show in the appendix the following formula:

$$P_a^\mu = \rho\sqrt{-g} \left[\left(-e + \frac{\partial e}{\partial\alpha} \right) \delta_\rho^\mu + \frac{\partial e}{\partial\beta} \tilde{S}_\rho^\mu + \frac{\partial e}{\partial\theta} \widetilde{S}_\lambda^\mu \tilde{S}_\rho^\lambda \right] (K^{-1})^{\rho\nu} \gamma_{ab} \xi^b{}_\nu, \tag{24}$$

where by $\widetilde{S}_\lambda^\mu \tilde{S}_\rho^\lambda$ we denote, as usual, the “tilde” part of the tensor $\tilde{S}_\lambda^\mu \tilde{S}_\rho^\lambda$. The above formula enables us to express the Piola–Kirchhoff tensor in terms of the fields and their derivatives if the function $e=e(\alpha, \beta, \theta)$ is known. In this way the dynamical equations (22) become second-order differential equations for the unknown fields ξ^a .

To analyze the physical meaning of these equations, we will rewrite them in the form of energy–momentum conservation,

$$\nabla_\mu t_\nu^\mu = 0, \tag{25}$$

where the energy–momentum tensor density is defined by the usual canonical formula,

$$t_\nu^\mu := P_a^\mu \xi^a{}_\nu - \delta_\nu^\mu \Lambda. \tag{26}$$

The conservation law (25) is implied by the dynamical equations (22) via the standard Noether procedure which we use in the sequel. The particular feature of the mechanics of continua formulated as a field theory is that eqs. (25) are simply equivalent to eqs. (22), even if (25) contains four equations while (22) contains only three equations. To prove this equivalence, we calculate the partial derivatives of the components of the energy–momentum tensor,

$$\begin{aligned} \partial_\mu t_\nu^\mu &= \xi^a{}_\nu \partial_\mu \frac{\partial\Lambda}{\partial\xi^a{}_\mu} + \frac{\partial\Lambda}{\partial\xi^a{}_\mu} \partial_\mu \xi^a{}_\nu - \partial_\nu \Lambda \\ &= \xi^a{}_\nu \partial_\mu \frac{\partial\Lambda}{\partial\xi^a{}_\mu} + \frac{\partial\Lambda}{\partial\xi^a{}_\mu} \partial_\mu \xi^a{}_\nu - \frac{\partial\Lambda}{\partial\xi^a} \xi^a{}_\nu - \frac{\partial\Lambda}{\partial\xi^a{}_\mu} \partial_\nu \xi^a{}_\mu - \frac{\partial\Lambda}{\partial g_{\rho\sigma}} \frac{\partial g_{\rho\sigma}}{\partial x^\nu}. \end{aligned}$$

Using the symmetry of the second derivatives of ξ^a we obtain

$$\partial_\mu t_\nu^\mu = \xi^a{}_\nu \left(\partial_\mu \frac{\partial\Lambda}{\partial\xi^a{}_\mu} - \frac{\partial\Lambda}{\partial\xi^a} \right) - \frac{\partial\Lambda}{\partial g_{\rho\sigma}} \frac{\partial g_{\rho\sigma}}{\partial x^\nu}.$$

This formula holds in any coordinate system x^μ . For a given point in space–time we can always choose a coordinate system such that the derivatives of the metric vanish at this particular point. In this particular system the partial derivatives become equal to covariant derivatives and the above equality reads

$$\nabla_\mu t^\mu_\nu = \left(\partial_\mu \frac{\partial A}{\partial \xi^a_\mu} - \frac{\partial A}{\partial \xi^a} \right) \xi^a_\nu. \tag{27}$$

But both sides of the above formula are covector densities. Hence, once proved in a particular coordinate system, the formula remains valid in any system due to its tensorial character. Observe that we proved eq. (27) using only the definition of the energy–momentum tensor, without assuming field equations. This means that (27) is an identity. It remains valid for any configuration $\xi^a = \xi^a(x^\mu)$. Taking into account the relation $u^\mu \xi^a_\mu = 0$, we see that the identity

$$u^\nu \nabla_\mu t^\mu_\nu = 0 \tag{28}$$

holds due to the definition of t^μ_ν in terms of the fields and their derivatives. Hence, only three equations among eqs. (25) are independent, and the maximal rank of the matrix (ξ^a_μ) implies the equivalence of the conservation laws (25) with the dynamical equations (21).

7. Stress–strain relations

Formulae (24) and (26) imply

$$t^\mu_\nu = \rho \sqrt{-g} \left[\left(-e + \frac{\partial e}{\partial \alpha} \right) \delta^\mu_\nu + \frac{\partial e}{\partial \beta} \tilde{S}^\mu_\beta + \frac{\partial e}{\partial \theta} \widetilde{S}^\mu_\lambda \widetilde{S}^\lambda_\rho \right] (K^{-1})^{\rho\sigma} \gamma_{ab} \xi^b_\sigma \xi^a_\nu - \delta^\mu_\nu A.$$

Using (11) we have

$$(K^{-1})^{\rho\sigma} \gamma_{ab} \xi^b_\sigma \xi^a_\nu = (K^{-1})^{\rho\sigma} (K_{\sigma\nu} + u_\sigma u_\nu) = \delta^\rho_\nu + u^\rho u_\nu = E^\rho_\nu. \tag{29}$$

Using also the identities $\tilde{S}^\mu_\rho E^\rho_\nu = \tilde{S}^\mu_\nu$ and $\widetilde{S}^\mu_\lambda \widetilde{S}^\lambda_\rho E^\rho_\nu = \widetilde{S}^\mu_\lambda \widetilde{S}^\lambda_\nu$, we obtain

$$\begin{aligned} t^\mu_\nu &= \rho \sqrt{-g} \left[\left(-e + \frac{\partial e}{\partial \alpha} \right) E^\mu_\nu + \frac{\partial e}{\partial \beta} \tilde{S}^\mu_\nu + \frac{\partial e}{\partial \theta} \widetilde{S}^\mu_\rho \widetilde{S}^\rho_\nu + \delta^\mu_\nu e \right] \\ &= \rho \sqrt{-g} \left(-e u^\mu u_\nu + \frac{\partial e}{\partial \alpha} E^\mu_\nu + \frac{\partial e}{\partial \beta} \tilde{S}^\mu_\nu + \frac{\partial e}{\partial \theta} \widetilde{S}^\mu_\rho \widetilde{S}^\rho_\nu \right). \end{aligned} \tag{30}$$

The energy–momentum tensor may therefore be decomposed into two parts, one parallel and the other orthogonal to the velocity field:

$$t^\mu_\nu = -\sqrt{-g} \epsilon u^\mu u_\nu - \tau^\mu_\nu,$$

where the orthogonal part is given by the formula

$$\tau_\nu^\mu := -\rho\sqrt{-g}\left(\frac{1}{\alpha}\frac{\partial e}{\partial\alpha}S_\rho^\sigma E_\nu^\mu + \frac{\partial e}{\partial\beta}\tilde{S}_\nu^\mu + \frac{\partial e}{\partial\theta}\widetilde{S_\rho^\mu S_\nu^\sigma}\right). \tag{31}$$

This part may be called the *relativistic stress tensor* or the *Cauchy tensor*. Equation (31) gives the stress tensor in terms of the strain tensor. The relation is universal and the response of the material to the deformation is uniquely given by the three coefficients: $\alpha^{-1}\partial e/\partial\alpha$, $\partial e/\partial\beta$ and $\partial e/\partial\theta$. For a general, non-linear material the coefficients depend on the deformation. In the particular case of the quasi-linear model the coefficients are constant:

$$\frac{1}{\alpha}\frac{\partial e}{\partial\alpha} = A, \quad \frac{\partial e}{\partial\beta} = B, \quad \frac{\partial e}{\partial\theta} = 0,$$

and the stress-strain relations become “quasi-linear”,

$$\tau_\nu^\mu = -\rho\sqrt{-g}(AE_\nu^\mu S_\rho^\sigma + B\tilde{S}_\nu^\mu).$$

Similarly as in the classical theory, the Cauchy tensor can be further decomposed into a part proportional to the “space projector” and a trace-free part,

$$\tau_\nu^\mu = -\sqrt{-g}(pE_\nu^\mu + Z_\nu^\mu),$$

where $p = -\rho \partial e/\partial\alpha$ is the pressure and the trace-free tensor

$$Z_\nu^\mu := -\rho\left(\frac{\partial e}{\partial\beta}\tilde{S}_\nu^\mu + \frac{\partial e}{\partial\theta}\widetilde{S_\rho^\mu S_\nu^\sigma}\right) \tag{32}$$

may be called the *relativistic deviatoric tensor*. Observe that, similarly to the classical theory, the pressure may be calculated as minus the derivative of the energy with respect to the specific volume $v = 1/\rho$ of the matter,

$$p = -\rho \frac{\partial e}{\partial\alpha} = -\rho \frac{\partial e}{\partial\rho} \frac{\partial\rho}{\partial\alpha} = \rho^2 \frac{\partial e}{\partial\rho} = -\frac{\partial e}{\partial v}.$$

We have finally ^{#4}

$$\begin{aligned} t_\nu^\mu &= -\sqrt{-g}(\epsilon u^\mu u_\nu + pE_\nu^\mu + Z_\nu^\mu) \\ &= -\sqrt{-g}[(\epsilon + p)u^\mu u_\nu + p\delta_\nu^\mu + Z_\nu^\mu]. \end{aligned} \tag{33}$$

The field equations (25) can therefore be rewritten in the following way:

$$\nabla_\mu [(\epsilon + p)u^\mu u_\nu + p\delta_\nu^\mu + Z_\nu^\mu] = 0,$$

or, equivalently

$$u_\nu \nabla_\mu [(\epsilon + p)u^\mu] + (\epsilon + p)u^\mu \nabla_\mu u_\nu = -\nabla_\nu p - \nabla_\mu Z_\nu^\mu. \tag{34}$$

^{#4} The sign of t_ν^μ has been chosen in such a way that $t_0^0 = P_a^0 \zeta^a_0 - A = H$ is the hamiltonian of the field. The identity $t^{\mu\nu} = -2 \partial A/\partial g_{\mu\nu}$ can be easily proved.

Due to the identity $u^\nu \nabla_\mu t_\nu^\mu = 0$ the above equation is equivalent with its part orthogonal to the velocity. Acting on both sides with the projector E we finally obtain the following equivalent version of the dynamical equations:

$$(\epsilon + p) u^\mu \nabla_\mu u_\nu = -E_\nu^\mu (\nabla_\mu p + \nabla_\rho Z_\mu^\rho). \quad (35)$$

We stress that the above equations contain relativistic hydrodynamics as a particular case, when the internal energy depends only on the matter density ρ or, equivalently, on α . In this case we have $\partial e / \partial \beta = \partial e / \partial \theta = 0$ and therefore $Z_\nu^\mu = 0$.

The interaction of the elastic material with other physical fields (electrodynamical, gravitational) may be obtained adding the corresponding lagrangian for the new field (together with the lagrangian of the interaction) to the mechanical lagrangian (20) and extending the variation to the degrees of freedom of the new field. In particular, we may treat the physical metric $g_{\mu\nu}$ as a dynamical object whose dynamics is governed by the Einstein equations. The complete theory can be obtained from the total lagrangian density, which is the sum of the Einstein–Hilbert lagrangian $-(1/8\pi)\sqrt{-g}R$ and the lagrangian (20). Due to gauge invariance of the theory the mechanical equations (25) are in this case implied by the Einstein equations via the Bianchi identities.

8. Remarks on the hamiltonian formulation of the dynamics

Having fixed the 3+1 decomposition of space–time we may formulate the dynamics of the field in terms of the initial value problem. The canonical variables of our theory are ξ^a (configuration) and P_a^0 (canonical momenta). The Poisson bracket between these quantities assumes its canonical (delta-like) form.

Similarly as in hydrodynamics, we may describe the Cauchy problem in terms of physical quantities (like, e.g., the density scalar ρ and the velocity components u^μ) rather than in terms of canonical variables. Expressing these quantities in terms of canonical configurations and momenta we may easily calculate the “non-canonical” Poisson bracket between them. The resulting formulae for the density and the velocities are obviously the same as in hydrodynamics (they have been calculated in ref. [15]). Unlike in hydrodynamics, the four objects ρ, u^k do not uniquely describe the gauge-independent part of the Cauchy data and two additional observables are necessary. It may be shown that the remaining parameters of the deformation (i.e., β and θ) can be used for this purpose. In this way we obtain an infinite-dimensional hamiltonian system with non-canonical symplectic structure, which will be discussed in a future paper.

9. Linear version of the theory

The linear approximation of the theory may be obtained assuming that the configuration of the material does not differ considerably from a given reference

configuration $\xi^a = x^a$. We introduce the *displacement field* ϕ which measures this difference,

$$\phi^a(x^\mu) = \xi^a(x^\mu) - x^a.$$

We assume also that the gravitational field is weak, i.e., $g_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu}$ is a small perturbation of the minkowskian flat metric $\eta_{\mu\nu}$. We treat therefore both ϕ^a and $f_{\mu\nu}$ as quantities of first order. For the sake of simplicity suppose that the material metric is flat, $\gamma_{ab} = \delta_{ab}$.

The theory has to be invariant with respect to gauge transformations $x^\mu \mapsto x^\mu - \psi^\mu$, where $\psi^\mu(x^\lambda)$ is any space-time vector field. The transformation acts in the following way on the dynamical objects of the theory:

$$\phi^a \mapsto \phi^a + \psi^a, \quad f_{\mu\nu} \mapsto f_{\mu\nu} + 2\psi_{(\mu\nu)} \tag{36}$$

(the symmetric part of the derivative of ψ corresponds to the Lie derivative of the metric η with respect to the gauge field ψ).

We begin calculating the first-order development of the matter current (3):

$$\begin{aligned} J^\mu &= \rho_0 \sqrt{\det \gamma} \epsilon^{\mu\nu\rho\sigma} \xi^1_{\nu} \xi^2_{\rho} \xi^3_{\sigma} \\ &= \rho_0 \epsilon^{\mu\nu\rho\sigma} (\delta^1_{\nu} + \phi^1_{\nu}) (\delta^2_{\rho} + \phi^2_{\rho}) (\delta^3_{\sigma} + \phi^3_{\sigma}) \\ &\simeq \rho_0 (\delta^{\mu}_0 + \phi^k_k \delta^{\mu}_0 - \dot{\phi}^i \delta^{\mu}_i), \end{aligned} \tag{37}$$

where the dot denotes the time derivative. Consequently

$$\begin{aligned} J_\mu &= (\eta_{\mu\nu} + f_{\mu\nu}) J^\nu \simeq \rho_0 (\eta_{\mu 0} + f_{\mu 0} + \phi^k_k \eta_{\mu 0} - \dot{\phi}^i \eta_{\mu i}), \\ J_\mu J^\mu &\simeq \rho_0^2 (-1 - 2\phi^k_k + f_{00}). \end{aligned}$$

Moreover

$$\begin{aligned} u^\mu &= \frac{1}{\sqrt{-J_\nu J^\nu}} J^\mu \simeq \frac{1}{\sqrt{1 + \phi^k_k - f_{00}}} (\delta^{\mu}_0 + \phi^k_k \delta^{\mu}_0 - \dot{\phi}^i \delta^{\mu}_i) \\ &\simeq \delta^{\mu}_0 + \frac{1}{2} f_{00} \delta^{\mu}_0 - \dot{\phi}^i \delta^{\mu}_i. \end{aligned} \tag{38}$$

The covariant version of this relation reads

$$u_\nu = (\eta_{\mu\nu} + f_{\mu\nu}) u^\mu \simeq \eta_{\nu 0} + \frac{1}{2} f_{00} \eta_{\nu 0} - \dot{\phi}^i \eta_{\nu i} + f_{\nu 0},$$

and therefore the first-order development of the projector U is given by

$$u_\nu u^\mu = \delta^{\mu}_0 \eta_{\nu 0} + f_{00} \delta^{\mu}_0 \eta_{\nu 0} - \dot{\phi}^i (\delta^{\mu}_0 \eta_{\nu i} + \delta^{\mu}_i \eta_{\nu i}) + \delta^{\mu}_0 f_{\nu 0},$$

or

$$u^\mu u_\nu = \begin{pmatrix} -1 & \dot{\phi}^i \\ f_{0j} - \dot{\phi}_j & 0 \end{pmatrix}. \tag{39}$$

To calculate the linear approximation of the strain tensor we observe that

$$S = -\frac{1}{2} \log K = -\frac{1}{2} (K - I) + O(\phi^3),$$

or

$$S_\nu^\mu \simeq -\frac{1}{2} (h_\nu^\mu - u^\mu u_\nu - \delta_\nu^\mu).$$

But, in the linear approximation, we have

$$\begin{aligned} h_{\lambda\nu} &= \delta_{ab} \delta_\lambda^a \delta_\nu^b + \delta_{ab} (\phi^a{}_\lambda \delta_\nu^b + \phi^b{}_\nu \delta_\lambda^a), \\ h_\nu^\mu &= (\eta^{\mu\lambda} - f^{\mu\lambda}) h_{\lambda\nu} \\ &= \delta_{ab} \delta_\mu^a \eta^{b\nu} + \delta_{ab} \phi^a{}_\mu \eta^{b\nu} + \delta_{ab} \phi^b{}_\lambda \delta_\mu^a \eta^{\lambda\nu} - f^{\nu b} \delta_{ab} \delta_\mu^a, \end{aligned}$$

or, equivalently,

$$h_\nu^\mu = \begin{pmatrix} 0 & \dot{\phi}^i \\ f_{0j} - \dot{\phi}_j & \delta_j^i + 2\eta^{ik} \phi_{(kj)} - f_j^i \end{pmatrix}.$$

Using (39) we finally obtain that only the space-space components of the strain tensor do not vanish, i.e., $S_\mu^0 = 0 = S_0^\mu$ and

$$S_j^i = \frac{1}{2} f_j^i - \eta^{ik} \phi_{(kj)}, \quad i, j, k = 1, 2, 3.$$

We stress that the above quantity is invariant with respect to the gauge transformations (36). Consequently, also the strain invariants α and β are gauge independent,

$$\begin{aligned} \alpha &= S_\mu^\mu = S_k^k = \frac{1}{2} f_k^k - \phi^k{}_k, \\ \beta &= \frac{1}{2} \tilde{S}^{\mu\nu} \tilde{S}_{\mu\nu} = \frac{1}{2} \tilde{S}^{kl} \tilde{S}_{kl} \\ &= \frac{1}{2} [(\frac{1}{2} f^{kl} - \phi^{(kl)}) (\frac{1}{2} f_{kl} - \phi_{(kl)}) - \frac{1}{3} (\frac{1}{2} f_k^k - \phi^k{}_k)^2]. \end{aligned}$$

Taking the Taylor expansion of u_I up to second-order terms in the strains we obtain the quasi-linear model,

$$\begin{aligned} u_I &\simeq \frac{1}{2} A \alpha^2 + B \beta \\ &\simeq \frac{1}{2} A (\frac{1}{2} f_k^k - \phi^k{}_k)^2 + \frac{1}{2} B [(\frac{1}{2} f^{kl} - \phi^{(kl)}) (\frac{1}{2} f_{kl} - \phi_{(kl)}) - \frac{1}{3} (\frac{1}{2} f_k^k - \phi^k{}_k)^2], \end{aligned}$$

with coefficients A and B defined by the derivatives of u_I at zero. The corresponding expressions for the matter density and the pressure read

$$\begin{aligned} \rho &= \rho_0 e^{-\alpha} \simeq \rho_0 (1 + \phi^k{}_k - \frac{1}{2} f_k^k), \\ p &= -\rho (\partial u_I / \partial \alpha) = -\rho A \alpha \simeq \rho_0 A (\phi^k{}_k - \frac{1}{2} f_k^k). \end{aligned} \quad (40)$$

To calculate the deviatoric tensor

$$Z_\nu^\mu = -\rho B \tilde{S}_\nu^\mu \simeq -\rho_0 B (S_\nu^\mu - \frac{1}{3} E_\nu^\mu \alpha),$$

observe that it is sufficient to take the zeroth-order expression for the projector E

since α is already of first order. We have

$$E_\nu^\mu = \delta_\nu^\mu + \delta_0^\mu \eta_{\nu 0} + O(1) . \tag{41}$$

Finally, only the space–space components of Z_μ^μ do not vanish,

$$Z^{kl} = -B\rho_0 [\frac{1}{2} f^{kl} - \phi^{(kl)} - \frac{1}{3} \eta^{kl} (\frac{1}{2} f^m_m - \phi^m_m)] . \tag{42}$$

The linear version of eqs. (35) reads:

$$\rho_0 m \delta_0^\mu \nabla_\mu u_\nu = - (\delta_\nu^\mu + \delta_0^\mu \eta_{0\nu}) (\partial_\mu p + \partial_\rho Z_\mu^\rho) \tag{43}$$

(covariant derivatives of first-order objects have been replaced by partial derivatives since the Christoffel symbols are of first order). To calculate the covariant derivative of u_ν we need the first-order approximation of the Christoffel symbols,

$$2\Gamma_{\mu\nu}^\alpha = \eta^{\alpha\beta} (\partial_\mu f_{\nu\beta} + \partial_\nu f_{\mu\beta} + \partial_\beta f_{\mu\nu}) .$$

Hence,

$$\begin{aligned} \delta_0^\mu \nabla_\mu u_\nu &= \partial_0 (\frac{1}{2} f_{00} - \dot{\phi}' \eta_{\nu l} + f_{0\nu}) - \Gamma_{0\nu}^\alpha \eta_{0\alpha} \\ &= \frac{1}{2} \dot{f}_{00} \eta_{0\nu} - \dot{\phi}'' \eta_{\nu l} + \dot{f}_{0\nu} - \frac{1}{2} \partial_\nu f_{00} . \end{aligned}$$

The above expression vanishes for $\nu=0$ [in agreement with the vanishing of the right hand side of (43)], and therefore the dynamical equations reduce to

$$m\rho_0 (-\dot{\phi}'' \eta_{kl} + \dot{f}_{0k} - \frac{1}{2} \partial_k f_{00}) = -\partial_k p - \partial_l Z_l^k .$$

Using the explicit formulae (40) and (42) for p and Z we finally obtain

$$\begin{aligned} m (\dot{\phi}''_k - \dot{f}_{0k} + \frac{1}{2} \partial_k f_{00}) &= - (A - \frac{1}{3} B) \partial_k (\frac{1}{2} f^j_j - \phi^j_j) \\ &\quad - B \partial_i (\frac{1}{2} f^i_k - \eta^{ij} \phi_{(jk)}) . \end{aligned} \tag{44}$$

We stress that both sides of the equation are gauge invariant. The right hand side can be called the elastic force. It reduces to the classical elastic force when the space–time metric is flat ($f_{\mu\nu}=0$). The term $m(\dot{f}_{0k} - \frac{1}{2}\partial_k f_{00})$ can be interpreted as the gravitational force and eq. (44) is the Newton equation.

The present formulation of the relativistic elasticity theory was conceived in 1988, when the first author (J.K.) was lecturing the mechanics of continua in Italy, at the University of Milan [19,20]. Both authors are very grateful to Professor Luigi Galgani, who was the spiritual father of this adventure.

Appendix. Proof of the stress–strain relation formula

The lagrangian density (20) depends on the derivatives of the fields only via the components of the tensor K . Therefore

$$P_a{}^\mu = \frac{\partial \mathcal{A}}{\partial \xi^a{}_\mu} = \frac{\partial \mathcal{A}}{\partial K_{\rho\nu}} \frac{\partial K_{\rho\nu}}{\partial \xi^a{}_\mu} = -\sqrt{-g} \frac{\partial \epsilon}{\partial K_{\rho\nu}} \frac{\partial K_{\rho\nu}}{\partial \xi^a{}_\mu}.$$

But

$$\frac{\partial K_{\rho\nu}}{\partial \xi^a{}_\mu} = \gamma_{ab} \xi^b{}_\rho \delta^\mu{}_\nu + \gamma_{ab} \xi^b{}_\nu \delta^\mu{}_\rho - \frac{\partial u_\rho}{\partial \xi^a{}_\mu} u_\nu - \frac{\partial u_\nu}{\partial \xi^a{}_\mu} u_\rho,$$

and consequently

$$\begin{aligned} P_a{}^\mu &= -2\sqrt{-g} \left(\frac{\partial \epsilon}{\partial K_{\rho\mu}} \gamma_{ab} \xi^b{}_\rho - \frac{\partial \epsilon}{\partial K_{\rho\nu}} \frac{\partial u_\nu}{\partial \xi^a{}_\mu} u_\rho \right) \\ &= -2\sqrt{-g} \left(N^{\mu\nu} \gamma_{ab} \xi^b{}_\nu - N^{\rho\nu} u_\rho \frac{\partial u_\nu}{\partial \xi^a{}_\mu} \right), \end{aligned} \tag{45}$$

where we denoted

$$N^{\mu\nu} := \frac{\partial \epsilon}{\partial K_{\mu\nu}} = e \frac{\partial \rho}{\partial K_{\mu\nu}} + \rho \left(\frac{\partial e}{\partial \alpha} \frac{\partial \alpha}{\partial K_{\mu\nu}} + \frac{\partial e}{\partial \beta} \frac{\partial \beta}{\partial K_{\mu\nu}} + \frac{\partial e}{\partial \theta} \frac{\partial \theta}{\partial K_{\mu\nu}} \right). \tag{46}$$

To calculate the derivatives of the invariants of the tensor $S = -\frac{1}{2} \log K$ with respect to the components of K we use the following formula of operator calculus:

$$\frac{\partial}{\partial K^\mu{}_\nu} \text{Tr} f(K) = (f'(K))^\nu{}_\mu,$$

where f is a differentiable function of a real variable, f' its derivative and where $f(K)$ denotes the operator-valued function of the operator K . Together with the definitions (17) the above formula implies

$$\begin{aligned} \partial \alpha / \partial K_{\mu\nu} &= -\frac{1}{2} (K^{-1})^{\mu\nu}, \\ \partial \beta / \partial K_{\mu\nu} &= -\frac{1}{2} \widetilde{S}_\rho{}^\nu (K^{-1})^{\rho\mu}, \\ \partial \theta / \partial K_{\mu\nu} &= -\frac{1}{2} \widetilde{S}_\rho{}^\nu \widetilde{S}_\lambda{}^\rho (K^{-1})^{\lambda\mu}. \end{aligned} \tag{47}$$

Moreover, formula (18) implies

$$\partial \rho / \partial K_{\mu\nu} = \rho_0 \partial e^{-\alpha} / \partial K_{\mu\nu} = \frac{1}{2} \rho (K^{-1})^{\mu\nu}.$$

Hence,

$$N^{\mu\nu} = \frac{\rho}{2} \left[\left(e - \frac{\partial e}{\partial \alpha} \right) (K^{-1})^{\mu\nu} - \frac{\partial e}{\partial \beta} \widetilde{S}_\rho{}^\mu (K^{-1})^{\rho\nu} - \frac{\partial e}{\partial \theta} \widetilde{S}_\rho{}^\mu \widetilde{S}_\lambda{}^\rho (K^{-1})^{\lambda\nu} \right]. \tag{48}$$

Using the relations $\widetilde{S}_\nu{}^\mu u^\nu = 0$, $\widetilde{S}_\rho{}^\mu \widetilde{S}_\nu{}^\rho u^\nu = 0$ and $(K^{-1})^\mu{}_\nu u^\nu = u^\mu$ we obtain

$$\begin{aligned}
 N^{\nu\rho}u_\rho \frac{\partial u_\nu}{\partial \xi^a{}_\mu} &= \frac{\rho}{2} \left(e - \frac{\partial e}{\partial \alpha} \right) u^\nu \frac{\partial u_\nu}{\partial \xi^a{}_\mu} \\
 &= \frac{\rho}{2} \left(e - \frac{\partial e}{\partial \alpha} \right) \frac{1}{2} \frac{\partial (u^\nu u_\nu)}{\partial \xi^a{}_\mu} = 0.
 \end{aligned} \tag{49}$$

Substituting (48) and (49) in (45) we finally obtain formula (24).

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